ON CERTAIN METHODS OF THE GEOMETRY OF POSITION.*

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I. INTRODUCTION.

THE present tendency of scientific specialization is generally justified; it is in the interest of pure science. This is especially true of the devolopment of mathematical branches, where everything foreign to the fundamental axioms is carefully discarded. A science in which this spirit is applied intentionally and systematically becomes a branch of philosophy; for instance, Grassmann's Linende Ansdehnmgslehre, Lobatschesky's Geometry, v. Standt's Geometrie der Lage, Weierstrasse's Theory of Functions, etc. As such they are of the greatest importance for the rigorous development of mathematical thought, and their value cannot be overestimated. It is a question, however, whether so-called pure methods are always what they pretend to be, and whether they are always to be recommended for pedagogical purposes. To take an example: Is it well to consider certain configurations in space in order to simplify the demonstration of propositions in plane geometry; or is it necessary, in order to be consistent, to apply only previously known propositions of plane geometry?

There is no doubt in my mind that the first can be done in a successful and consistent manner. In this paper I shall attempt to show the value of such methods for the teaching of the geometry of position. At this point I desire to say that descriptive geometry, as well as projective geometry, or the geometry of position, ought to be made regular courses in the mathematical departments of real universities. A knowledge of elementary descriptive geometry gives the student an invaluable power for the mastery of the more difficult problems of projective geometry and of higher geometry in general.† In what follows I shall assume the knowledge of ordinary descriptive geometry.

II. ADVANCED PLANE GEOMETRY.

There are a great number of propositions in plane geometry which appear in their natural light when considered as projections of figures in space. As such they are independent of metrical relations and it is unnatural to prove them by equations. As an example, I may mention the homology of triangles.

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†Regular courses in descriptive and projective geometry are now offered at nearly all universities of continental Europe.

Two triangles, A B C and A′B′C′, are homologous,

1. When the rays AA′, BB′, CC′, joining corresponding points, are concurrent; or, also,

2. When the points of intersection of corresponding sides A B and A′B′, BC and B′C′, CA and C′A′ are collinear.

Each of these two definitions as a hypothesis necessitates the other as a thesis.

Casey, in his Sequel to Euclid, which is very rich in beautiful examples and propositions, but without an organism, reduces advanced geometry to an incoherent mass of metrical facts. To prove the propositions concerning homologous triangles, he introduces ratios of areas of triangles, and applies the theorem of Menelaos concerning transversals.* The same method is followed in most treatises on plane geometry. The immortal elements of Euclid, which in themselves are of rare beauty and rigor, lead very soon to sterility when applied to projective properties of figures.

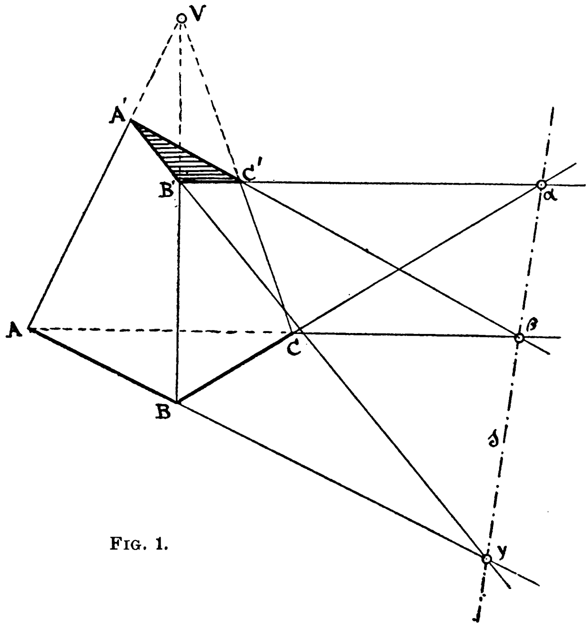

FIG. 1.

Homologous triangles appear in the simplest manner as plane sections of triangular pyramids. Let V be the vertex of such a pyramid and ABC and A′B′C′ the intersections of its edges, with two oblique planes, P and P′, respectively. Let S be the line of intersection of P and P′. A glance at the figure (fig. 1) shows that AB and A′B′, BC and B′C′, CA and C′A′ meet in points of S.

* Loc. cit., book sixth.

Now, two triangles, ABC and A′B′C′, in which AA′, BB′, CC′ produced are concurrent may always be considered as the projection of a triangular pyramid cut by two oblique planes in ABC and A′B′C′. Thus the foregoing proposition is established. In a similar manner the converse proposition may be proved.

As a second example, let us take the proposition concerning three circles in a plane:

The external centers of similitude of three circles of a plane are collinear. Any two internal centers of similitude are always collinear with one of the external centers.

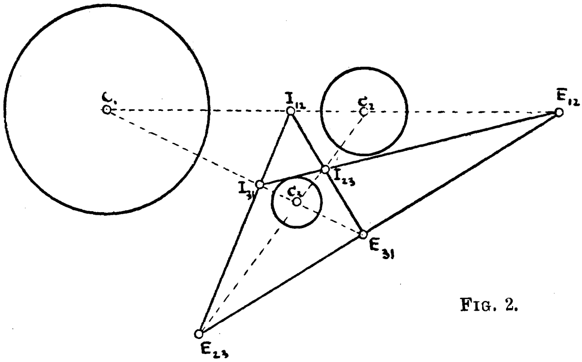

FIG. 2.

To prove this, let us consider three spheres whose projections are the three given circles in the plane. Any two of these spheres admit of an internal and external common tangent cone. Thus, designating the centers of the spheres by C_1, C_2, C_3, we obtain three external and three internal tangent cones whose respective vertices E_{12}, E_{23}, E_{31}, and I_{12}, I_{23}, I_{31}, are coplanar, the plane P passing through C_1, C_2, C_3. The three spheres and the six cones have the same common tangent planes and these are 2 by 2 symmetrical with respect to P. Any two symmetrical planes (there are eight common tangent planes) are common tangent planes to the three spheres and to three of the tangent cones.

The three vertices of these cones are therefore necessarily collinear. We have therefore the result that the six vertices of the common tangent cones are coplanar and are 3 by 3 situated in straight lines; *i. e.*, they form a complete quadrilateral.

The collinear groups are, fig. 2:

E_{12}	E_{12}	E_{23}	E_{31}
E_{23}	I_{23}	I_{31}	I_{12}
E_{31}	I_{31}	I_{12}	I_{23}

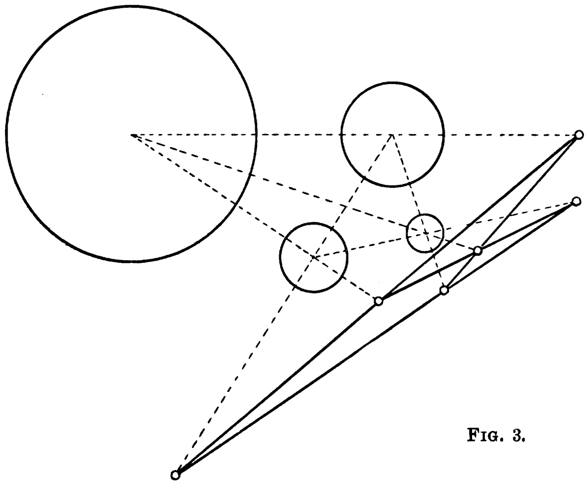

FIG. 3.

Any orthographic or central projection of this configuration leads immediately to the original proposition. We may conversely use relations in a plane to establish propositions in space. Take, for instance, four circles in a plane and construct their external centers of similitude (fig. 3). It is found that they form a complete quadrilateral, whose six points are 3 by 3 collinear. The four circles may be considered as projections of spheres in space. Here the collinearity of the vertices of corresponding common tangent cones still exists, and we have therefore the theorem:

The six external centers of similitude of any four spheres in space are coplanar.

Similar propositions hold for the internal centers.

III. ORTHOGRAPHIC PROJECTION.

In most text-books on descriptive geometry no attention is paid to certain geometrical principles which, as it will appear, form the base for nearly all projective constructions. It is also remarkable how easy these principles and relative propositions may be derived from exact intuition in space. From this point of view the necessity of including certain geometrical propositions in a course on descriptive geometry is imperative. I shall illustrate the points in question by the treatment of affinity of figures* (homology with an infinite center).

* Fiedler: Geometrie der Lage, vol. 1, pp. 1-115.

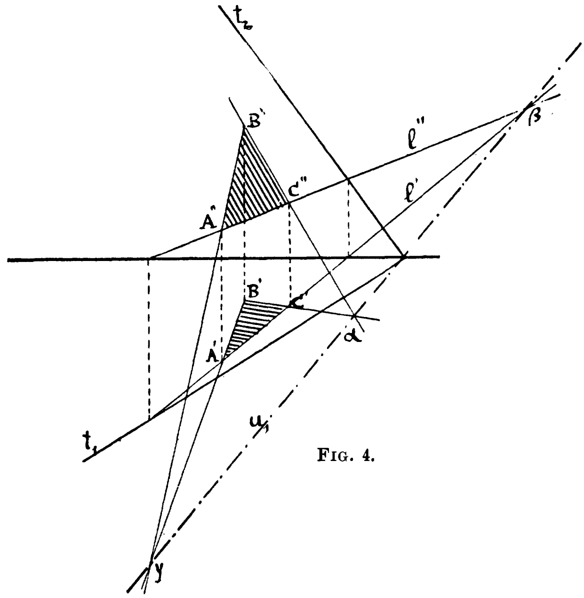

FIG. 4.

The horizontal and vertical projections of any point in the bisecting plane of the second and fourth angle coincide. From this it follows that the projections of the piercing point of any line with the bisecting plane, which shall be designated by U, coincide, and are consequently obtained as the point of intersection of the two projections of the straight line. Let l' and l'' be the projections of a line l and D_1 their point of intersection. In a similar manner, we designate by A_1, B_1, C_1, ... ; a_1, b_1, c_1, ... the coinciding projections of points A, B, C, ... and lines a, b, c, ... in U. As two points determine a straight line and as the projections of a point in U coincide, it follows that the projections of any line in U coincide also. Any plane P with the traces t_1 and t_2 intersects U in a line u, whose projections coincide in u_1. Evidently the traces t_1, t_2 and the lines u and u_1 meet in the same point T of the ground line. The line u is therefore determined if another point is known. To find such a point, assume any straight line l in the plane P, figure 4, and construct its intersection L with U. The straight line connecting T with L, or in the projections T with L_1, is the required line. Every line in the plane P intersects the line u; hence its projections meet in a point of the line u_1. From this it is seen that the projections of points, lines and figures in a plane P are related by the two laws:

1. Corresponding projections of lines meet in points of a fixed straight line (axis of affinity).

2. Corresponding projections of points are situated in parallel lines.

These are precisely the laws dominating the affinity of figures in a plane, which itself is a special case of homology (when the center is infinitely distant). The propositions which we have established in connection with homologous triangles may be specialized for triangles related by affinity and their proof does not present the slightest difficulty. In fact, affinity results from homology by considering a triangular prism instead of a triangular pyramid, and it is clear that our previous reasoning applies also to this case.

I shall now mention two metrical specializations of affinity. Assume two triangles ABC and A'B'C' related by affinity and assume that the parallel lines AA', BB', CC' be perpendicular to the axis of affinity, then the ratios between the distances of corresponding points from the axis of affinity are equal and the proposition holds:

The areas of the two triangles are to each other as the distances of corresponding points from the axis of affinity.

If this ratio is unity then their areas are equal and the triangles are in axial symmetry.

If AA', BB', CC' are parallel to the axis of affinity then the areas of ABC and A'B'C' are equal.

This is a case which is never considered in plane geometry.

I might extend this subject still further, but I hope the previous treatment will be sufficient to show in what a simple and effective manner, and without losing much time, important geometrical propositions may be obtained from the study of elementary descriptive geometry. It is hardly necessary to point out that these propositions are conversely the most efficient and rapid means to make projections of plane figures.

Suppose, for instance, that the horizontal projection of a plane figure A, B, C, ..., the vertical projection C'' of C and the line of intersection u of the plane of ABC... with U be given. The vertical projection A''B''C''... may be constructed by the previous principles alone. Thus, to find B'', connect B'C' and prolong to the intersection with u_1; join the latter point to C'', and from B' draw a perpendicular to the ground line. Where this perpendicular intersects the last line, is the required vertical projection B'' of B. By this method three lines are sufficient to find a required point, while the ordinary method by means of the traces of the plane requires four or five (two parallels and two perpendiculars in one case, and two connecting lines and three perpendiculars in the other). The same principle may be applied to find the true shape of a plane figure from its projections; three lines give a point of the rabatted figure. This method is, therefore, also in the line of Lemoin's Géométrographie,* where fig-

*M. E. Lemoin: Principes de la Geometrographie, Archiv der Mathematik und Physik, vol. 1, 1901, p. 99.

ures are constructed and investigated with reference to the greatest attainable simplicity. I can say from experience that there is hardly a subject in descriptive and advanced plane geometry in which the student takes more real interest than in this method of presenting the construction of plane figures in descriptive geometry, and of introducing propositions of projective geometry. It seems to me the most natural way to higher geometry.

IV. COLLINEATION.

As in the previous chapter, I shall start from elementary constructions in descriptive geometry and through the study of perspective gradually arrive at the most general expression of collineation.

A central projection, or a perspective, is determined by the plane of projection (pictorial plane) and the center (eye). Assuming the plane of the paper as the plane of projection and any point in space as the center, it is possible to construct the perspective of any figure in space in this plane. The center can be most easily located by a circle in the plane of projection. The radius of this circle is the distance of the center from the plane and the center of the circle is the orthographic projection of the center of projection upon the plane of projection. This circle has been introduced into geometry by Professor Fiedler, of Zurich, and he calls it distance circle (distanzkreis).*

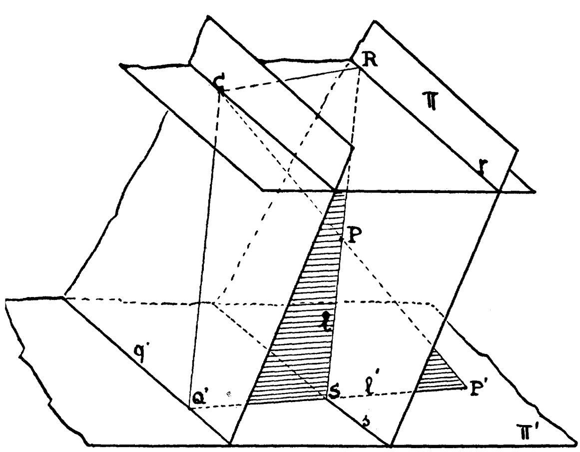

FIG. 5.

Let II′ be the plane of projection and II an arbitrary plane, whose projection upon II′ shall be made from a center C. Let S be the line of intersection of II′ and II, fig. 5. To obtain the projection of any point P′ of any point P in II, connect P with the center C; then the point of intersection of this connecting line (indefinitely produced)

*Darstellende Geometrie, vol. I, 1883. It must be mentioned that Courinery already uses a "cercle a distance" in his Geometrie Perspective, Paris, 1828.

with II′ is the required point P′. In a similar manner the projection l′ of a line l in II is obtained as the line of intersection of the plane passing through C and l with II′. From this construction the following fundamental laws are immediately clear, and are in fact a mere restatement of the laws of homologous triangles previously considered.

To every point of II corresponds one and only one point of II′, and conversely, and both points are on the same ray through C.

To every straight line of II corresponds a straight line of II′, and conversely; and both lines meet at the same point of s (holds for any line).

To the infinite line q of II corresponds a line q′ of II′, which is parallel to s. Conversely, to the infinite line r′ of II′ corresponds a line r parallel to s.

The plane II is usually determined by its trace s in II′ and either of the lines r and q′. If a straight line l in II is given, intersecting the trace s in S′, the corresponding line l′ is obtained by drawing a line through C parallel to l and marking its point of intersection Q′ with q′. It is evident that Q′ is the projection of the infinitive point of l, and the projection of l consequently passes through S′ and Q′. Another way is to produce l till it intersects r in R and to join C with R. The line through S parallel to CR is the required projection l′ of l. From the figure, it is seen that CRSQ′ is a parallelogram and that PS : PR = P′S : CR.

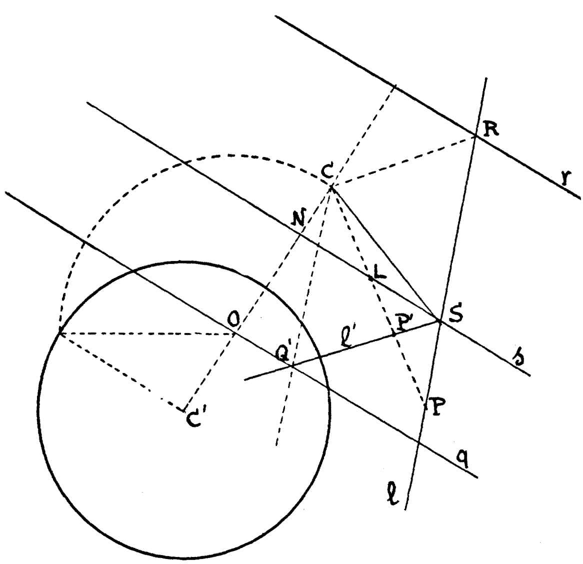

Fig. 6.

The planes through C parallel to II and II′ form a space of a parallelepipedon. Keeping II′ fixed it is possible by rotations about s and

q′ as axes to rabatte the planes through C, and II into II′ without changing the figures situated in these planes. After the motion there is still CR = and ∥ Q′S and SP′ = SP′, so that the distances PR and PS remain also unchanged. From this it follows that after the motion P′ and the rabatted position of P lie on a ray through the revolved position of C. The laws expressing the relations between the revolved and projected figure are therefore the same as those between the figure in space (II) and its projection II′. After the rabattement, fig. 5 assumes the form of fig. 6. Here l and l′ are the two corresponding lines which with s and SC form a pencil of four rays through S. As CP and CQ intersect this pencil there is

$$(CLP'P) = (CMQ'Q).$$

The value of (CMQ′Q) is $\frac{CQ'}{MQ'} = \frac{CO}{NO} = k$, ray; *i. e.*, entirely independent of the position of l, l′, and CP. Thus, drawing any ray through C and intersecting s in S′, any two points P and P′ on this ray of the central projection form a constant ratio with C and S′. For all possible pairs P and P′ of corresponding points

$$(CSPP') = \text{constant}.$$

The different cases of central projection may be classified according to the position of the center of projection and the value of the constant k.*

Laying any Carterian system of coordinates through C, and designating the coordinates of any pair of corresponding points P and P′ by x, y, and x′, y′, respectively, it is an easy matter to derive from the figure the general form of the relation existing between these coordinates:

$$\left.\begin{aligned} x' &= \frac{ax}{dx+ey+f}, \\ y' &= \frac{ay}{dx+ey+f}, \end{aligned}\right\} \qquad (1)$$

where $\frac{f}{a}$ is the constant k of the projection.

Formulas (1) represent the transformation of a point P into P′, called perspective.

The equation of the line r is $dx + ey + f = o$, and its corresponding line r′ is infinitely distant. To the line q ($x = \infty$, $y = \infty$) corresponds the line q′ with the equation $dx' + ey' - a = o$. The axis s is obtained by putting $x' = x$, $y' = y$, and its equation is $dx + ey + f - a = o$. Comparing the equations of the lines r, s, q′, and their distances from C, it is found that

$$\frac{f-a}{\sqrt{d^2+e^2}} = \frac{f}{\sqrt{d^2+e^2}} - \frac{a}{\sqrt{d^2+e^2}},$$

*Fiedler, loc. cit., p. 95.

as it also appears from the figure. The deduction of formulas (1) from an actual perspective construction has the great advantage that all results gained from the analytical discussion may easily be interpreted constructively and geometrically. It is noticed that a perspective transformation depends upon three essential constants, *i. e.*, if its center is fixed. This is equivalent to saying that the axis of perspective can be chosen in a doubly infinite number of ways, and the constant k in a singly infinite number. If to the perspective transformation P, as given by (1), we apply a dilation D, defined by

$$x'' = dx', \quad y'' = y', \tag{2}$$

and which may be considered as a special case of perspective where the center is infinite in a direction perpendicular to s. From this it is seen that the combination of a perspective and dilation (PD) may be expressed by

$$\left.\begin{aligned} x'' &= \frac{ax'}{dx' + ey' + f}, \\ y'' &= \frac{by}{dx' + ey' + f}. \end{aligned}\right\} \tag{3}$$

Applying to this transformation consecutively a transformation by equal areas (A), defined by

$$\left.\begin{aligned} x''' &= x'' + ky'' \\ y''' &= y'' \end{aligned}\right\}, \tag{4}$$

and which may also be considered as a special case of perspective, in which C is infinitely distant in a plane perpendicular to the bisecting plane of II and II′; then a transformation (T), defined by

$$\left.\begin{aligned} x^{(4)} &= x''' + p \\ y^{(4)} &= y''' + q \end{aligned}\right\}, \tag{5}$$

and finally a rotation (R),

$$\left.\begin{aligned} x^{(5)} &= nx^{(4)} - my^{(4)} \\ y^{(5)} &= nx^{(4)} + my^{(4)} \end{aligned}\right\}, \tag{6}$$

$$(m^2 + n^2 = 1),$$

we arrive at a transformation of the form

$$\left.\begin{aligned} x' &= \frac{ax + by + c}{dx + ey + f}, \\ y' &= \frac{gx + hy + j}{dx + ey + f}. \end{aligned}\right\} \tag{7}$$

This transformation is characterized by eight independent constants, and is called, as is well known, a projective transformation, or collineation. It may be considered as the result of the combined transformations

$$(P\,D\,A\,T\,R), \tag{8}$$

which are determined by 3, 1, 1, 2, 1 parameters, respectively. It is

also well known that the latter are subgroups of the general projective group. From the constructive study of collineation we have thus arrived at the conception of the continuous projective groups of transformation. The method which we have followed makes it again possible to follow the train of reasoning in the discussion of groups by illustrative constructions. It may, of course, be extended to space.

V. CONCLUSION.

A well-arranged parallelism of descriptive, synthetic and analytic methods in organic connection, a method chiefly cultivated by Fiedler, seems to be most valuable for a rapid introduction into the fields of higher geometry. The introduction of critical discussion concerning the foundations of geometry into elementary treatises has a tendency to confuse the student. The establishment of the fundamental principles of projective geometry independent of metrical relations or of the eleventh axiom of Euclid may follow an introduction as outlined in this paper. v. Standt's construction, Fiedler's projective coordinates, Caley's and Klein's absolute geometry, or non-Euclidian geometry, must form indispensable parts of such an advanced study. In the method followed by us, and which is partly, also, that of Poncelet, Steiner, and Charles, the projective properties of the circle are easily established and transferred to conics by perspective. It is, however, necessary to show that all curves of the second order defined as products of projective ranges and pencils, or analytically by equations of the second degree, are conics. There is no difficulty in doing this.

Descriptive analytic methods are also of invaluable service for the study of congruences and complexes of rays and for higher geometry in general. In this respect I may mention the treatment of linear complexes, the congruence of bisecants of a twisted cubic, of the "Null system," by descriptive methods, and their elegant representation by certain partial differential equations.*

There is one branch of mathematics which is rarely mentioned in connection with projective geometry, namely, kinematics. In the hands of Penucellier, Kempe, Sylvester, Hart, and, in recent times, especially by Professor Koenigs, of Paris, kinematics has rendered valuable services to modern geometry. Starting from the beautiful theorem † that every plane and twisted algebraic curve and every algebraic surface may be described by a linkage, Koenigs invented a planigraph, and quite recently, also, a link-motion perspectivograph, realizing collineation. A short treatment of these interesting linkages would form a valuable addition to any text-book on projective geometry.

* See my paper "On the Congruences of Rays (3, 1) and (1, 3)," Annals of Mathematics; also, S. Lie: Geometrie der Beruhrungs-transformationen, vol. I, p. 326.

† Koenigs: Lecons de Cinematique, Paris, 1897, pp. 271, 297, 305.